

# Dual of Big-bang and Big-crunch

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Starting from the Janus solution and its gauge theory dual, we obtain the dual gauge theory description of the cosmological solution by the procedure of double analytic continuation. The coupling is driven either to zero or to infinity at the big-bang and big-crunch singularities, which are shown to be related by the S-duality symmetry. In the dual Yang-Mills theory description, these are non singular at all as the coupling goes to zero in the  $\mathcal{N}=4$  Super Yang-Mills theory. The cosmological singularities simply signal the failure of the supergravity description of the full type IIB superstring theory.

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# 1 Introduction

The duality between gravity and gauge theory seems the key to understand the quantum gravity. Since the original AdS/CFT correspondence[1, 2, 3, 4], there have appeared many variants of the gravity/gauge theory correspondence. However, until now, there are no concrete examples of the gauge theory dual to the cosmological evolution of gravity theory.

As the cosmology includes big-bang and, possibly, big crunch singularities, it would be helpful to understand their nature once we identify a gauge theory dual of cosmological evolution.

Our route to a gauge theory dual of cosmological evolution is rather straightforward. For this, we take the Janus solution[5] as a starting point. It is a controlled nonsupersymmetric deformation of  $AdS_5 \times S^5$  keeping  $SO(3, 2) \times SO(6)$  part of  $SO(4, 2) \times SO(6)$  global symmetries, which turns out to be stable[5, 6, 7, 8]. The five dimensional geometry is  $AdS_4$  sliced while keeping  $S^5$  part intact. The dilaton runs through the bulk as function of the slicing coordinate  $\mu$  and the asymptotic AdS boundary at the ends of the range  $\mu$  has two separate parts joined at their boundaries. The values of dilaton on the two parts become different and the dual Yang-Mills (YM) theory has a Janusian nature. Namely the dual YM theory has two domains separated by a codimension one interface, where the YM coupling jumps from one to the other[5]. More precise version of the Janus dual YM theory is proposed in Ref. [9]. The operator  $\mathcal{L}'_0$  dual to the dilaton is the variant of the  $\mathcal{N}=4$   $SU(N)$  SYM Lagrange density in which the scalar kinetic term is  $X^I \partial^2 X^I / 2$  rather than  $-\partial X^I \partial X^I / 2$ . The action density for the Janus dual is given by  $\mathcal{L}'_0$  multiplied by the inverse coupling squared which takes different value in each domain. There have been further developments in the studies of Janus type solutions[10, 11, 12, 13], its related issues[7, 14, 15] and other types of dilatonic deformation[16, 17, 18, 19, 20, 21, 22].

The standard dictionary of AdS/CFT correspondence says that the on-shell supergravity action gives the generating functional of connected graphs of the dual field theories, which has validity in the large  $N$  limit. Of course the supergravity should be replaced by the full string theory for the finite  $N$  and string coupling. The generating functional has to be renormalized in order to get finite correlation functions. Thus the generating functional is defined with help of a renormalization group (RG) scale plus the coupling at the RG scale. The bulk direction may be interpreted as a direction of the RG scale and, at a finite RG scale, the coupling varies spatially in a smooth manner.

We note then the time dependent branch of solution preserving  $SO(4, 1) \times SO(6)$  symmetries can be obtained by a specified procedure of double analytic continuation<sup>1</sup>. The supergravity solution

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<sup>1</sup>It was recently noted in Ref. [23] that there is a general correspondence between domain wall and cosmological

then obtained is a Friedmann-Robertson-Walker (FRW) type cosmology exhibiting the big-bang and big-crunch singularities. We define the field theory dual for the cosmological solution by the procedure of double analytic continuation of the renormalized generating functional. If one has full nonperturbative knowledge including all order  $1/N$  corrections of the Janusian gauge theory, the procedure of the double analytic continuation can be made precise, leading, in principle, to the exact field theory dual of the cosmology.

We carry out the double analytic continuation of the generating functional in its leading order. The big-bang and big-crunch turns out to be related by S-duality of IIB string theory and these singularities arise because the coupling goes either to zero or to infinity in the gravity description. We will see, however, that this is simply a failure of the supergravity description and nothing singular happens in the  $\mathcal{N}=4$  SYM theory in its weak coupling limit.

The organization of our paper is as follows. In section 2, we will review the Janus solution and its gauge theory dual. In section 3, we will discuss the meaning of the bulk geometry of the Janus solution in relation with renormalization. In section 4, we will obtain the cosmological solution by the procedure of the double analytic continuation. In section 5, we will present the dual gauge theory interpretation of the cosmological solution including the big-bang and big-crunch singularities. We will end with a discussion of the other conformal compactification, where the scale change becomes the time evolution after the double analytic continuation.

## 2 Janus solution and its CFT dual

The Janus solution[5] corresponds the stable nonsupersymmetric dilatonic deformation of  $AdS_5 \times S^5$  which preserves  $SO(3,2) \times SO(6)$  part of the original  $SO(4,2) \times SO(6)$  global symmetries. The dilaton is tuned on additionally, while the original five-form field strength is modified minimally.

Following Ref. [5], let us review the set of coordinate systems for the  $AdS_d$  space, which may be useful for our application. The  $AdS_d$  space is defined by a hyperboloid in  $R^{2,d-1}$

$$-X_0^2 - X_d^2 + X_1^2 + \cdots + X_{d-1}^2 = -1 . \quad (1)$$

The global coordinate covers the entire region of the  $AdS_d$  space. Using the parameterization,

$$X_0 = \frac{\cos \tau}{\cos \theta}, \quad X_d = \frac{\sin \tau}{\cos \theta}, \quad X_i = \tan \theta \, n_i, \quad i = 1, \cdots, d-1 , \quad (2)$$

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solutions.

with the unit vector  $n_i$  in  $R^{d-1}$ , the metric on the global  $AdS_d$  is given by

$$ds^2_{AdS_d} = \frac{1}{\cos^2 \theta} ( -d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2 ) , \quad (3)$$

where  $\theta \in [0, \pi/2]$ .

The Poincaré patch is another standard parameterization of AdS space, which is dual to the  $\mathcal{N}=4$  SYM theory on  $R^{1,3}$  for the case of  $AdS_5 \times S^5$ . By parameterization,  $X_0 = \frac{1}{2} (z + (1 + \vec{x}^2 - t^2)/z)$ ,  $X_d = t/z$ ,  $X_{i=1,\dots,d-2} = x_i/z$ , and  $X_{d-1} = \frac{1}{2} (z - (1 - \vec{x}^2 + t^2)/z)$ , the metric takes the form

$$ds^2_{AdS_d} = \frac{1}{z^2} ( -dt^2 + d\vec{x}^2 + dz^2 ) , \quad (4)$$

where  $\vec{x} = (x_1, \dots, x_{d-2})$  and  $z \in [0, \infty]$ .

For the description of the Janus solution,  $AdS_{d-1}$  slicing of the  $AdS_d$  space is very useful. We use  $X_{d-1} = w$  as one coordinate ranged over  $[-\infty, +\infty]$  and any coordinate system of the  $AdS_{d-1}$  space of radius  $\sqrt{1+w^2}$  for the rest. The metric takes then the following form

$$ds^2_{AdS_d} = \frac{dw^2}{1+w^2} + (1+w^2) ds^2_{AdS_{d-1}} . \quad (5)$$

Defining a new coordinate  $\mu$  by  $w = \tan \mu$ , the metric is rewritten as

$$ds^2_{AdS_d} = f_0(\mu) (d\mu^2 + ds^2_{AdS_{d-1}}) , \quad (6)$$

where  $f_0(\mu) = 1/\cos^2 \mu$  and  $\mu \in [-\pi/2, \pi/2]$ .

The conformal boundary of the global AdS metric (3) is located at  $\theta = \pi/2$  and has the shape of  $R \times S^3$  where  $R$  is the time direction, while that of the Poincaré patch (4) is at  $z = 0$  and of the shape  $R^{1,3}$ . In the  $AdS_{d-1}$  slicing (6), the appearance of the boundary is less trivial although there is no change of its shape.

First consider the case where the global coordinate is used for the  $AdS_{d-1}$  slice in (6). Note then the metric can be written as

$$ds^2_{AdS_d} = \frac{1}{\cos^2 \mu \cos^2 \lambda} ( -d\tau^2 + \cos^2 \lambda d\mu^2 + d\lambda^2 + \sin^2 \lambda d\Omega_{d-3}^2 ) , \quad (7)$$

with  $\lambda \in [0, \pi/2]$ . The constant time section of this metric is conformal to a half of  $S^{d-1}$ , since the range of  $\mu$  is just from  $-\pi/2$  to  $\pi/2$ . If  $\mu$  had ranged over  $[-\pi, \pi]$ , it would have been the full sphere. The boundary consists of two parts, one of which is at  $\mu = -\pi/2$  and the other at  $\mu = \pi/2$ . These two parts are joined through a surface,  $\lambda = \pi/2$ , which is a codimension one interface in the boundary. The  $\mu = \pi/2$  (or  $\mu = -\pi/2$ ) part is a half of  $S^{d-2}$  as  $\lambda$  ranged over  $[0, \pi/2]$ , so over all

the boundary makes up the full  $S^{d-2}$ . For the Janus solution, the range of  $\mu$  becomes elongated, but the structure of the conformal boundary remains to be the same  $S^{d-2}$ .

Next one may take the Poincaré patch for the  $AdS_{d-1}$  slice in (6). Then the metric takes the form

$$ds^2_{AdS_d} = \frac{1}{y^2 \cos^2 \mu} (-dt^2 + d\vec{x}_{d-3}^2 + dy^2 + y^2 d\mu^2) , \quad (8)$$

with  $y \in [0, \infty]$ . One can easily see that, by the change of coordinate  $x = y \sin \mu$  and  $z = y \cos \mu$ , the above metric turns into the conventional form of the Poincaré patch AdS,

$$ds^2_{AdS_d} = \frac{1}{z^2} (-dt^2 + d\vec{x}_{d-3}^2 + dx^2 + dz^2) . \quad (9)$$

Again the boundary consists of two parts, one of which is at  $\mu = \pi/2$  and the other at  $\mu = -\pi/2$ . These two parts are joined through a codimension one surface,  $y = 0$ , forming a  $d - 2$  dimensional flat Euclidean space, or  $d - 1$  dimensional Minkowski space when including the time. For the Janus solution, the structure of the conformal boundary remains again the same, on which the dual YM theory is defined.

The ansatz for the Janus solution is given by

$$\begin{aligned} ds^2 &= f(\mu) (d\mu^2 + ds^2_{AdS_4}) + ds^2_{S^5} , \\ \phi &= \phi(\mu) , \\ F_5 &= 2f(\mu)^{\frac{5}{2}} d\mu \wedge \omega_{AdS_4} + 2\omega_{S^5} , \end{aligned} \quad (10)$$

where  $\omega_{AdS_4}$  and  $\omega_{S^5}$  are the unit volume forms on  $AdS_4$  and  $S^5$  respectively. Thus, in particular, the five sphere  $S^5$  remains unchanged keeping the  $SO(6)$  R-symmetry. But the supersymmetry is completely broken. The  $SO(3, 2)$  of  $AdS_4$  is clearly preserved as one can see from the above ansatz.

The relevant IIB supergravity equations of motion are given by

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi - \frac{1}{96} F_\alpha^{\mu\nu\lambda\delta} F_{\beta\mu\nu\lambda\delta} &= 0 , \\ \partial_\alpha (\sqrt{g} g^{\alpha\beta} \partial_\beta \phi) &= 0 , \\ *F_5 &= F_5 , \end{aligned} \quad (11)$$

together with the Bianchi identity  $dF_5 = 0$ . The equation of motion for the dilaton can be integrated leading to

$$\phi'(\mu) = \frac{c}{f^{\frac{3}{2}}(\mu)} . \quad (12)$$

The Einstein equations give rise to

$$\begin{aligned} 2f'f' - 2ff'' &= -4f^3 + \frac{c^2}{2} \frac{1}{f} , \\ 12f^2 + f'f' + 2ff'' &= 16f^3 . \end{aligned} \quad (13)$$

It is easy to see that these equations are equivalent to the first order differential equation

$$f' f' = 4f^3 - 4f^2 + \frac{c^2}{6} \frac{1}{f} , \quad (14)$$

corresponding to the motion of a particle with zero energy in a potential given by

$$V(f) = -4 \left( f^3 - f^2 + \frac{c^2}{24} \frac{1}{f} \right) . \quad (15)$$

The solution,  $f(\mu, c)$ , may be expressed analytically in terms of the Weierstrass  $\mathcal{P}$ -function[11].

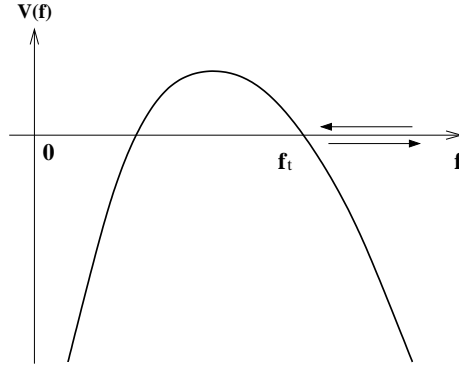


Figure 1: The dynamics corresponds to the particle motion under a potential with zero energy. The trajectory of our concern is the one in which the particle starts from infinity, reflected at  $f_t$  and goes back to infinity.

Note that  $c = 0$  corresponds to the unperturbed  $AdS_5 \times S^5$  with a constant dilaton. For  $c \in [0, \frac{9}{4\sqrt{2}}]$ , the potential has two positive zeros and, as depicted in Figure 1, we are particularly interested in the trajectory starting from infinity, reflected and going back to infinity. Note that  $\mu$  is ranged over  $[-\mu_0, \mu_0]$  with

$$\mu_0 = \int_{f_t}^{\infty} \frac{df}{2\sqrt{f^3 - f^2 + \frac{c^2}{24} \frac{1}{f}}} \geq \frac{\pi}{2} , \quad (16)$$

where  $f_t$  is the turning point of the potential. The dilaton in this case varies over a finite range and the string couplings can be made arbitrarily small;

$$\phi(\mu_0) - \phi(-\mu_0) = c \int_{-\mu_0}^{\mu_0} \frac{d\mu}{f^{\frac{3}{2}}(\mu)} = c \int_{f_t}^{\infty} \frac{df}{f^{\frac{3}{2}} \sqrt{f^3 - f^2 + \frac{c^2}{24} \frac{1}{f}}} . \quad (17)$$

Adopting the global coordinate for the  $AdS_4$  slice as in (7), the metric in this case becomes

$$ds^2 = \frac{f(\mu)}{\cos^2 \lambda} ( - d\tau^2 + \cos^2 \lambda d\mu^2 + d\lambda^2 + \sin^2 \lambda d\Omega_2^2 ) . \quad (18)$$

The spatial section of the conformal metric, i.e. the metric inside the parenthesis is depicted in Figure 2. Only the surface of  $\mu$  and  $\lambda$  coordinates is drawn, where each point represents  $S^2$ . The boundary consists of two parts; one is at  $\mu = -\mu_0$  and the other at  $\mu = \mu_0$ . These two halves of  $S^3$  are joined through the north and south poles. The dilaton varies from one constant,  $\phi_- = \phi(-\mu_0)$ , at one half of the boundary at  $\mu = -\mu_0$ , to another,  $\phi_+ = \phi(\mu_0)$ , at the other half of the boundary at  $\mu = \mu_0$ , running through the bulk as  $\mu$  changes.

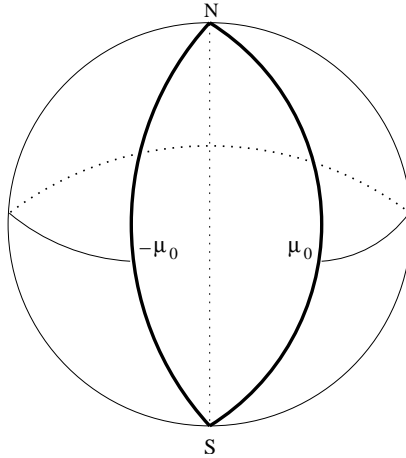


Figure 2: The conformal diagram of the constant time slice is depicted here. Only the surface of  $\mu$  and  $\lambda$  coordinates is shown, where each point corresponds to  $S^2$ .

In the pure AdS case, the geometry covers only a half of the surface of the globe with the range of  $\mu \in [-\pi/2, \pi/2]$ . As one can see from (16) for the Janus solution, the range of  $\mu$  becomes larger as  $c$  grows.

When we adopt the Poincaré patch for the  $AdS_4$  slice as in (8), the metric is written as

$$ds^2 = \frac{f(\mu)}{y^2} (-dt^2 + d\vec{x}^2 + dy^2 + y^2 d\mu^2) . \quad (19)$$

The conformal mapping of the spatial section is depicted in Figure 3. The bulk of the Janus solution corresponds to the region under the solid line. Each point on the plane represents  $R^2$ . As in the pure AdS case,  $y = \infty$  corresponds to the horizon. Again the boundary is at  $\mu = \pm\mu_0$ . Each of these is a half of  $R^3$ , being joined together through the wedge  $W$  that is  $R^2$ .

Again the dilaton varies from one constant,  $\phi_- = \phi(-\mu_0)$ , at one half of the boundary  $\mu = -\mu_0$ , to another,  $\phi_+ = \phi(\mu_0)$ , at the other half  $\mu = \mu_0$ , running through the bulk as  $\mu$  increases. Hence from the viewpoint of the boundary, the dilaton is constant in each half, taking the value of  $\phi_+$  and  $\phi_-$  respectively. We introduce a boundary coordinate  $x_3$  by  $\pm y$  for  $\mu = \pm\mu_0$ . The discontinuity of the dilaton occurs through the joint  $W$ . One might worry about possible singularities around the

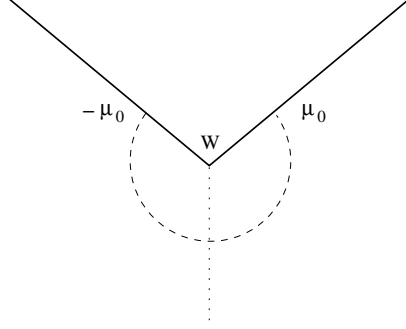


Figure 3: The conformal diagram of the spatial section of the Poincarè type coordinate is depicted here. The bulk of the Janus solution corresponds to the region under the solid line. Each point on the plane corresponds to  $R^2$ . The total boundary again consists of two parts, each of which has the geometry of a half of  $R^3$ . These two parts are joined through  $W$ , which is a codimension one interface.

wedge  $W$ . However, this is just an artifact of the conformal diagram. Nothing singular happens around the wedge from the viewpoint of the full geometry.

Near the boundary  $\mu \rightarrow \pm\mu_0$ , one can easily show that the factor  $f$  and the dilaton behave as

$$f = \frac{1}{(\mu \mp \mu_0)^2} + O(1) , \quad \phi = \phi_{\pm} \mp \frac{c}{4}(\mu \mp \mu_0)^4 + O((\mu \mp \mu_0)^6) . \quad (20)$$

In the standard AdS/CFT dictionary the asymptotic value of the dilaton is identified with the coupling constant of the four dimensional  $\mathcal{N}=4$  SYM theory living on the boundary. The subleading term is identified with the expectation value for the operator  $\text{Tr}(F^2)$ . Hence the interpretation of our solution is that the boundary consists of two half spaces, given by  $\mu = \pm\mu_0$ , where the coupling constant is given by  $g^2 = 4\pi e^{\phi_{\pm}}$  respectively. This is the holographic dual suggested in Ref. [5].

More precise version of the holographic dual was identified in Ref. [9]. Note first that the boundary metric of the dual CFT is obtained by multiplying the bulk metric by  $h^2$  where  $h$  has a linear zero at the boundary. Different choice of the factor  $h$  leads to the different boundary metric. The standard dictionary of AdS/CFT correspondence then says that the near boundary behaviors of the bulk field,  $\phi_{\Delta}$ , dual to the operator  $O_{\Delta}$  of dimension  $\Delta$  can be expressed in terms of  $h$  by

$$\phi_{\Delta} \sim a_h(x)h^{4-\Delta} + b_h(x)h^{\Delta} + \dots \quad (21)$$

where  $x$  is the boundary coordinate. Then this corresponds to turning on the source  $\int dx a_h(x)O_{\Delta}(x)$  while the one point function is given by  $\langle O_{\Delta}(x) \rangle = -(2\Delta - 4)b_h(x)$ . By the choice of  $h = y/\sqrt{f}$ , the boundary metric becomes flat Minkowski  $R^{1,3}$ . The operator dual to  $\phi$  is the  $\mathcal{N}=4$  SYM Lagrange density but the precise form is more subtle as noted in Ref. [9]. The dual operator  $\mathcal{L}'_0$  is the variant of the  $\mathcal{N}=4$  Lagrange density in which the scalar kinetic term is  $X^I \partial^2 X^I / 2$  rather than



$-\partial X^I \partial X^I / 2$ . The action for the Janus dual is given by

$$S = \int d^4x \frac{1}{g^2(x_3)} \mathcal{L}'_0 \quad (22)$$

where

$$\frac{1}{g^2(x_3)} = \frac{1}{g_+^2} \Theta(x_3) + \frac{1}{g_-^2} \Theta(-x_3) = \frac{1}{\bar{g}^2} (1 - \gamma \epsilon(x_3)) \quad (23)$$

with  $\gamma = (g_+^2 - g_-^2)/(g_+^2 + g_-^2)$  and  $\bar{g}^2 = (g_+^2 + g_-^2)/(2g_+^2 g_-^2)$ . Here  $\Theta(x)$  is the step function and  $\epsilon(x)$  for the sign function. The supergravity prediction with precise coefficient[24, 9] is given by

$$e^\phi = e^{\phi_\pm} \left(1 - \frac{2\pi^2}{N^2} \langle \mathcal{L}' \rangle h^4 + \dots\right) \quad (24)$$

where  $\mathcal{L}' = \mathcal{L}'_0/\bar{g}^2$ . Therefore the expectation value is

$$\langle \mathcal{L}' \rangle = \epsilon(x_3) \frac{N^2}{2\pi^2} \frac{c}{4x_3^4}. \quad (25)$$

Then one can confirm the above gravity prediction on the one point function using the conformal perturbation theory in the gauge theory side[9].

Finally if  $c$  exceeds  $c_{cr} = 9/(4\sqrt{2})$ , the zero energy motion of particle reaches the point  $f = 0$ , where the geometry develops a naked singularity. The singularity is timelike here. As one approaches the singularity, the magnitude of dilaton diverges. Hence the string theory becomes either extremely weakly coupled or strongly coupled. The dual field theory description is not identified in this regime. But we note that the diverging dilaton is to do with the appearance of the curvature singularity. Since the geometry involves the timelike naked singularity, whether this regime is physical or not is unclear.

### 3 Meaning of the bulk geometry

The bulk in the AdS/CFT correspondence may be interpreted as the space of RG scale in CFT side[25]. The gravity on-shell action is identified with the generating functional of the connected graphs of the boundary quantum field theory. The correlation functions computed this way exhibit divergences due to the infinite volume and this is the place where the renormalization becomes relevant. The flow into the bulk direction may be interpreted as the RG flows as we change the cut-off scale of the dual gauge theory. By the Legendre transform of the generating functional, one gets the quantum effective action defined at a given cut-off scale.

In the Janus gauge theory, the YM coupling abruptly changes from  $g_+$  to  $g_-$ , which is encoded at the boundary values  $\phi(\pm\mu_0)$ . For the definition of the boundary field theory, one needs only

information of  $\phi(\mu)$  at the boundary since the boundary field theory is defined at infinite cut-off scale in the momentum space.

What is the meaning of the precise form of  $\phi$  as a function of  $\mu$  in the Janusian case? We would like argue below that this bulk running of the dilaton can have its meaning if one lowers the cut-off scale to a finite value.

Following the prescription of Ref. [25], the dilaton at a finite cut-off here may be interpreted as a renormalized coupling at a given RG scale. In order to describe the details of the RG flow, the Fefferman-Graham coordinate, as adopted in Ref. [25], is most convenient. Unfortunately, for the case of the Janus geometry, the Fefferman-Graham coordinate does not cover the whole region of the bulk[7]. Rather it covers only the region of  $\mu_L \leq \mu \leq \mu_0$  with nonvanishing  $\mu_L$  excluding some bulk part around the wedge of Figure 3. Thus in this coordinate, the study of renormalization at finite cut-off scale is problematic. For this reason, to carry out the holographic renormalization, one has to develop a new technique with generic lapse and shift functions. This is certainly interesting direction for a further study but beyond the scope of this note. For our present purpose, we shall work in a particular metric that can cover the whole of the Janus geometry:

$$ds^2 = N^2 d(\Lambda^{-1})^2 + g_{ij}(dx^i + N^i d(\Lambda^{-1}))(dx^j + N^j d(\Lambda^{-1})), \quad (26)$$

where we take

$$g_{ij}dx^i dx^j = \Lambda^2(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) \quad (27)$$

with an appropriate choice of  $\Lambda$ ,  $x_3$ ,  $N$  and  $N^i$ . Since  $\Lambda$  is responsible for the scaling of  $g_{ij}$ , it can be interpreted as a physical cut-off scale coordinate[25].

Comparing (26), (27), and (19), one finds that the cut-off scale coordinate  $\Lambda$  is  $\sqrt{f(\mu)}/y$ . Then along constant  $\Lambda$ ,  $x_3$  coordinate is given by

$$dx_3^2 = dy^2 + y^2 d\mu^2 = \left(f^2 + \frac{c^2}{24f^2}\right) \frac{d\mu^2}{\Lambda^2} \quad (28)$$

such that

$$ds^2 = \Lambda^2(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2), \quad (29)$$

for constant  $\Lambda$  slice. Explicitly  $x_3$  reads

$$x_3 = \frac{1}{\Lambda} \int_0^\mu d\mu \sqrt{f^2 + \frac{c^2}{24f^2}} \equiv \frac{1}{\Lambda} G(\mu). \quad (30)$$

For  $c = 0$ ,  $f_0 = 1/\cos^2 \mu$  and  $1/\Lambda = y \cos \mu$ .  $G(\mu)$  can be obtained as  $G = \tan \mu$  and, then,  $x_3 = y \sin \mu$ . The full metric in this case becomes the standard form of the Poincaré metric,

$$ds^2 = \Lambda^2(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + (d\Lambda^{-1})^2). \quad (31)$$

One can express  $\phi(\mu)$  as a function of  $\Lambda$  and  $x_3$  by

$$\phi(\mu) = \phi(G^{-1}(\Lambda x_3)). \quad (32)$$

We interpret this  $e^{\phi(x_3)}$  as the  $x_3$  dependent coupling squared seen at the scale  $\Lambda$ , which becomes now a smooth function of  $x_3$  at any finite  $\Lambda$ .

## 4 Cosmological Solutions

In this section we like to discuss about the cosmological solution as a dilatonic deformation of  $AdS_5$  geometry. The solution involves big-bang and big-crunch singularities and how to interpret these in terms of dual field theory is our main concern.

The FRW type cosmological solution can be obtained from the Janus solution by performing the double analytic continuation<sup>2</sup>;

$$X_0 \rightarrow -iX_{d-1}, \quad X_{d-1} \rightarrow iX_0 \quad (33)$$

We note that the ansatz in (10) for the Janus solution can be rewritten as

$$\begin{aligned} ds^2 &= \frac{f(w)}{1+w^2} ds_{AdS_5}^2 + ds_{S^5}^2, \\ \phi &= \phi(w), \\ F_5 &= 2 \left( \frac{f(w)}{1+w^2} \right)^{\frac{5}{2}} \omega_{AdS_5} + 2\omega_{S^5}, \end{aligned} \quad (34)$$

where  $\omega_{AdS_5}$  is the unit volume form on  $AdS_5$ . Then by the above double analytic continuation, the ansatz becomes

$$\begin{aligned} ds^2 &= \frac{f(X_0)}{1-X_0^2} ds_{AdS_5}^2 + ds_{S^5}^2, \\ \phi &= \phi(X_0), \\ F_5 &= 2 \left( \frac{f(X_0)}{1-X_0^2} \right)^{\frac{5}{2}} \omega_{AdS_5} + 2\omega_{S^5}, \end{aligned} \quad (35)$$

which is valid only for  $|X_0| \leq 1$ . With the transformation of

$$c \rightarrow -ic, \quad (36)$$

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<sup>2</sup>This is not the conventional Wick rotation of quantum field theories. For instance, the ranges of  $\mu$  and  $\alpha$  below differ from each other. Here it is rather a formal way of obtaining new solutions and field theories.

the dilaton equation is solved by

$$(1 - X_0^2)\phi'(X_0) = \frac{c}{f^{\frac{3}{2}}(X_0)} . \quad (37)$$

Also the Einstein equations are reduced to

$$(1 - X_0^2)^2(f'(X_0))^2 = -4f^3 + 4f^2 + \frac{c^2}{6} \frac{1}{f} . \quad (38)$$

Before going into details, let us briefly comment on the case of  $X_0^2 > 1$ . This case cannot be obtained from the Janus solution by the double analytic continuation. But one can compute equations of motion directly starting from the ansatz,

$$\begin{aligned} ds^2 &= \frac{f(X_0)}{X_0^2 - 1} ds_{AdS_5}^2 + ds_{S^5}^2 , \\ \phi &= \phi(X_0) , \\ F_5 &= 2 \left( \frac{f(X_0)}{X_0^2 - 1} \right)^{\frac{5}{2}} \omega_{AdS_5} + 2\omega_{S^5} . \end{aligned} \quad (39)$$

Then the type IIB equations of motion are reduced to

$$(X_0^2 - 1)\phi'(X_0) = \frac{c}{f^{3/2}} , \quad (40)$$

$$(X_0^2 - 1)^2(f'(X_0))^2 = 4f^2 + 4f^3 + \frac{c^2}{6} f^{-1} . \quad (41)$$

There is an extra change in the sign of  $f^3$  term, which is due to the change of signs in the term of five form squared.

Since the potential is negative definite, the particle hits the  $f = 0$  region inevitably. From the form of the scalar curvature,

$$R = -20 - \frac{c^2}{2f^4} , \quad (42)$$

one can see that the solution involves curvature singularities. The singularity is timelike as noted in Ref. [5] and nothing to do with the cosmological singularities. For this reason, we shall not discuss this branch of solution further here.

Now let us get back to the case of  $|X_0| < 1$ . By the change of variable  $X_0 = \tanh \alpha$ , the equation (38) becomes

$$\left( \frac{df}{d\alpha} \right)^2 = -4f^3 + 4f^2 + \frac{c^2}{6} \frac{1}{f} , \quad (43)$$

which can be again interpreted as a particle motion with a zero energy under a potential,

$$V(f) = 4 \left( f^3 - f^2 - \frac{c^2}{24} \frac{1}{f} \right) . \quad (44)$$

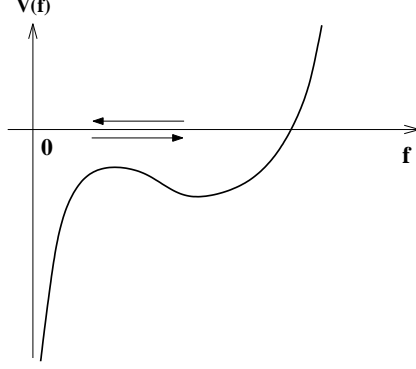


Figure 4: The dynamics again corresponds to the particle motion under a potential with zero energy. The particle starts from zero, reflected at the turning point and goes back to zero.

The shape of the potential is depicted in Figure 4.

For  $c = 0$ , the solution corresponds to the unperturbed  $AdS_5$  with  $f = 1 - X_0^2 = 1/\cosh^2 \alpha$ . The range of  $\alpha$  is given by  $\alpha \in [-\infty, \infty]$ .

If we turn on nonvanishing  $c$ , the range of  $\alpha$  becomes finite now. By choosing  $\alpha = 0$  at the turning point, the range for  $\alpha$  is given by  $[-\alpha_0, \alpha_0]$  where  $f(\pm\alpha_0) = 0$ . For small  $f$  region,  $V \sim -c^2/(6f)$  and the equation is approximated as

$$\frac{df}{d\alpha} \sim \pm \frac{|c|}{\sqrt{6f}}, \quad (45)$$

which is solved by

$$f^{\frac{3}{2}} \sim \frac{3|c|}{2\sqrt{6}} |\alpha \pm \alpha_0|. \quad (46)$$

Then the dilaton behaves as

$$\phi \sim \pm \frac{2\sqrt{6}}{3} \ln |\alpha \pm \alpha_0|. \quad (47)$$

Hence for positive/negative  $c$ , the dilaton starts from  $\mp\infty$ , monotonically increases/decreases and ends up with  $\pm\infty$ .

The scalar curvature in (42) become singular as  $\alpha \rightarrow \pm\alpha_0$ . These singularities are spacelike cosmological singularities. The precise form of the solution,  $f$ , can be obtained from the Janus solution,  $f_J(\mu, c)$ , by

$$f(\alpha, c) = f_J(i\alpha, -ic). \quad (48)$$

Introducing the new coordinate  $\tau$  by  $\sin \tau = \tanh \alpha$ , the five dimensional metric can be presented in the FRW form,

$$ds^2 = \frac{f(\tau)}{\cos^2 \tau} (-d\tau^2 + \cos^2 \tau ds_{EAdS_4}^2) , \quad (49)$$

where  $ds_{EAdS_4}^2$  is the metric of the Euclidean  $AdS_4$  space,

$$ds_{EAdS_4}^2 = \frac{dr^2}{1+r^2} + r^2 ds_{S^3}^2 = d\eta^2 + \sinh^2 \eta ds_{S^3}^2 . \quad (50)$$

Thus the solution respects  $SO(4, 1) \times SO(6)$  symmetries. The Penrose diagram for this cosmology is depicted in Figure 5.

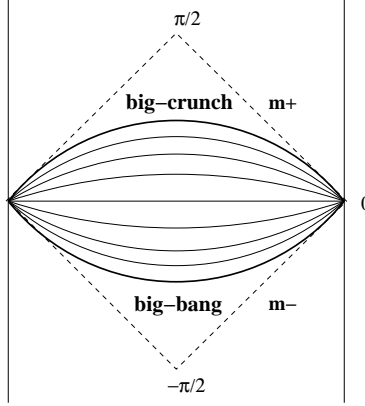


Figure 5: The Penrose diagram is depicted here for the dilatonic cosmological solution. The two parallel lines represent the boundary of original global  $AdS_5$  space. The upper thick solid curve corresponds to the big-crunch whereas the lower one to the big bang singularity.

If one uses the Poincaré patch coordinate for the Euclidean  $AdS_4$ , the metric can take the form,

$$ds^2 = \frac{f(\alpha)}{y^2} (-y^2 d\alpha^2 + dy^2 + dx_1^2 + dx_2^2 + dx_E^2) . \quad (51)$$

Unlike the case of the usual Poincaré patch and global coordinates of  $AdS_5$  [4], the Penrose diagram for (51) is the same as the one for the spacetime (49). Let us illustrate how this happens. We are interested in the conformal structure of  $(y, \alpha)$  directions, so a point in the Penrose diagram corresponds to a plane spanned by  $(x_1, x_2, x_E)$ . The conformal structure can be obtained from

$$-y^2 d\alpha^2 + dy^2 = -dt^2 + dy^2 , \quad (52)$$

with  $t = y \sinh \alpha$  and  $x = y \cosh \alpha$ . The coordinates  $\alpha$  and  $y$  are respectively ranged over  $[-\alpha_0, \alpha_0]$  and  $[0, \infty)$  and  $\alpha = \pm \alpha_0$  are the locations of the spacelike singularities. If  $\alpha$  were ranged over  $(-\infty, \infty)$ , then  $(y, \alpha)$  space covers the whole quadrant specified by  $x + t \geq 0$  and  $x - t \geq 0$ . The singularities are then obviously residing within the quadrant. We now conformally compactify the space  $(t, x)$  by

$$-dt^2 + dy^2 = -\sec^2 w_+ \sec^2 w_- dw_+ dw_- , \quad (53)$$

where  $\tan w_{\pm} = t \pm x$  with  $w_{\pm} \in [-\pi/2, \pi/2]$ . The boundary of the usual Poincaré patch is at  $w_+ - w_- = 0$  corresponding to the right vertical line of Figure 5. The quadrant in the above is now covered by  $0 \leq w_+ \leq \pi/2$  and  $-\pi/2 \leq w_- \leq 0$ . By carefully identifying locations of the singularities of  $\alpha = \pm\alpha_0$  within the quadrant, one finds the Penrose diagram of Figure 5.

## 5 Dual of Big-bang and Big-crunch

Our proposal for the dual of the cosmological solution is as follows. Since the cosmological solution can be obtained by the double analytic continuation described in the previous section, we propose that the corresponding dual gauge field theory can be again defined by the procedure of the double analytic continuation. But the coupling of the dual field theory is time dependent and this may interfere with the process of the renormalization. Because of this, the field theory may not be defined at infinite cut-off scale.

Thus we do not perform the double analytic continuation directly in the Janus dual gauge theory defined at the boundary. Instead, we note that there is a generating functional for the renormalized correlation functions at a given RG scale. Or alternatively one may talk about the quantum effective action by the Legendre transformation. We perform the double analytic continuation at the level of the generating functional. If one has full nonperturbative knowledge of the generating functional including all order  $1/N$  corrections, the procedure of the double analytic continuation can be made precise leading to the exact gauge theory dual of the cosmology. As we will see more below, the double analytic continuation for the gauge theory on (29) is given by

$$\begin{aligned} t &\rightarrow -ix_E, & x_3(\mu, c) &\rightarrow ix_0 = x_3(i\alpha, -ic) \\ \mu &\rightarrow i\alpha, & c &\rightarrow -ic, \end{aligned} \tag{54}$$

in the leading order supergravity approximation.

If one ignores the issue of renormalization, the quantum effective action corresponds roughly to the  $\mathcal{N}=4$  SYM Lagrangian density with time dependent YM coupling. At the big-bang or the big-crunch then, the YM coupling goes either to zero or to infinity. For both cases, the supergravity approximation breaks down. Namely for zero coupling limit, the curvature radius  $l = (g^2 N)^{\frac{1}{4}} l_s$  becomes zero and the  $\alpha'$  expansion breaks down completely. For the strong coupling limit, the string tree level approximation breaks down completely too. This way we may understand the appearance of the cosmological singularity, which signals simply the breaking down of the geometric description. We further note that the strong coupling limit is related to the weak coupling limit by the S-duality symmetry of the type IIB superstring theory. Thus one may see that in this correspondence, the

big-bang and big crunch are related by the S-duality of the  $\mathcal{N}=4$  SYM theory. Since the gauge theory is time dependent now, the in- and the out-vacuum will differ and there is the effect of particle production in general. Hence the S-duality does not mean that the in- and the out-vacuum are the same.

To see the indication of break down of the geometric description, we use the coordinate (51). The scale is then again identified with  $\Lambda = f^{\frac{1}{2}}/y$ . For the constant  $\Lambda$ , we defined  $x_0$  coordinate by

$$-dx_0^2 = dy^2 - y^2 d\alpha^2 = - \left( f^2 - \frac{c^2}{24f^2} \right) \frac{d\alpha^2}{\Lambda^2} \quad (55)$$

such that

$$ds^2 = \Lambda^2 (-dx_0^2 + dx_1^2 + dx_2^2 + dx_E^2), \quad (56)$$

for constant  $\Lambda$  surface. Then  $x_0$  is given by

$$x_0 = \frac{1}{\Lambda} \int_0^\alpha d\alpha \sqrt{f^2 - \frac{c^2}{24f^2}} \equiv \frac{1}{\Lambda} K(\alpha). \quad (57)$$

For  $c = 0$ ,  $f_0 = 1/\cosh^2 \alpha$  and  $1/\Lambda = y \cosh \alpha$ . The function  $K$  can be obtained as  $K = \tanh \alpha$  and, then,  $x_0 = y \sinh \alpha$ . The full metric for  $c = 0$  becomes the standard form of the Poincaré metric,

$$ds^2 = \Lambda^2 (-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + (d\Lambda^{-1})^2). \quad (58)$$

Note that one major difference from the Janusian case is that the boundary at infinity is no longer available meaning that it is now just a point in the Penrose diagram of Figure 5.

For non zero  $c$ , the definition of  $x_0$  breaks down at  $\alpha = \pm\alpha_H$  defined by

$$f(\pm\alpha_H) = \frac{\sqrt{|c|}}{(24)^{\frac{1}{4}}} \quad (59)$$

due to this signature change of the right hand side of (55). For  $\alpha_0 > |\alpha| > \alpha_H$ , the double analytic continuation also breaks down due to the signature problem.

Whether one can avoid this signature flip by the other choice of coordinate in the holographic renormalization is not entirely clear. Due to the presence of the singularities and the behavior of dilaton, the geometrical description will be breaking down in any ways and the dual gauge theory description can only be defined by the double analytic continuation of the correlators of the Janus gauge theory.



Since our proposed correspondence is not the direct relation between the geometry and the boundary gauge theory via the AdS/CFT dictionary, an intuition from geometry is limited anyway and we shall not further clarify meaning of the break-down in the above.

However, since in the weak coupling limit, the  $\mathcal{N}=4$  SYM perturbation theory is well defined clearly, we see that only the geometric description becomes problematic. We know clearly that the *classical*  $\mathcal{N}=4$  YM theory in the weak coupling limit is nothing like the ten dimensional supergravity. In this respect, the failure of description in this regime is rather obvious. Therefore, the time dependent field theory is well defined for all  $x_0$  and the appearance of the cosmological singularity is simply due to the failure of supergravity description.

Finally, with help of the function  $K$ ,  $\phi(\alpha)$  can be expressed as a function of  $\Lambda$  and  $x_0$  by

$$\phi(\alpha) = \phi(K^{-1}(\Lambda x_0)). \quad (60)$$

Then  $e^{\phi(\Lambda x_0)}$  can be interpreted as the  $x_0$  dependent YM coupling squared seen at scale  $\Lambda$ . We illustrate  $\phi(\Lambda x_0)$  in Figure 6 for  $c = 0.1$ . As a function of  $x_0$ , the shape of  $\phi$  simply scales as  $\Lambda$  varies, which is obvious from (60). The slope becomes steeper as  $\Lambda$  becomes larger.

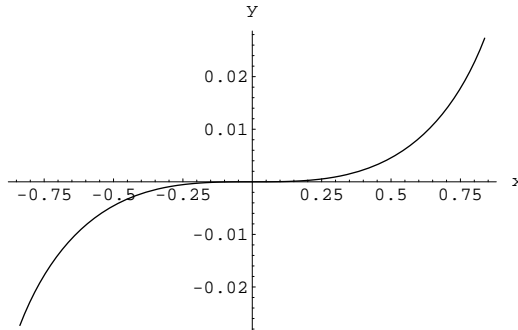


Figure 6: The behavior of the dilaton as a function of  $\Lambda x_0$  is drawn here for  $c = 0.1$ . The x-axis is for  $\Lambda x_0$  and the y-axis for  $\phi(\Lambda x_0) - \phi(0)$ . One finds that the maximum value of  $|x_0 \Lambda|$  is numerically 0.907 at the sign flip and the value of  $|\phi(\lambda x_0) - \phi(0)|$  there is 0.0423.

## 6 Discussion

In this note, we obtain the dual gauge field theory of cosmological solution by the procedure of double analytic continuation starting from the Janus solution and its gauge theory dual. We note that the big-bang and big-crunch singularities in this cosmology are resolved naturally in the dual gauge theory description. These cosmological singularities signal simply a failure of the supergravity description of the full IIB string theory. Our analysis of the double analytic continuation is based on

the supergravity approximation and there might be a chance to improve the analysis if part of the supersymmetries are preserved. Recently supersymmetric version of Janus solution[9, 10, 11, 12] has been investigated and this might be the place where one has a better control of corrections.

The conformal compactification of Janus solution leading to the boundary gauge theory on  $R \times S^3$  can be discussed similarly and we have not presented details of the computation. Note that there is yet another compactification of the Janus solution leading another boundary conformal field theory. Namely the conformal factor  $h$  can be chosen as  $1/\sqrt{f}$ . For this case as noted in [9], the boundary geometry is given by two  $AdS_4$  joined at their boundary  $R \times S^2$ . The change of cut-off scale then corresponds to the change of  $\mu$  as  $f$  is only function of  $\mu$ . As we are away from the boundary by changing  $f(\mu)$  ( $= f(-\mu)$ ) down to a finite value, the two  $AdS_4$  spaces at  $\mu$  and  $-\mu$  are joined at their boundary. Since the coupling is only function of the scale  $\mu$ , the coupling remains constant for each  $AdS_4$  while there is a jump through the common boundary. This is contrasted the case of the choice of  $h = y/\sqrt{f}$ , where the coupling shows again step function dependence of  $x_3$  at the UV limit. But, when we lower the cut-off scale to a finite value, the coupling varies smoothly as a function of  $x_3$ .

For the above  $AdS$  type compactification, what derives the change of coupling from the view point of dual gauge theory is not quite clear.

We now consider performing the double analytic continuation of this new compactification. The scale coordinate  $\mu$  turns into the time coordinate. The  $AdS_4$  becomes Euclidean  $AdS_4$  by the analytic continuation. Thus the scale change turns into the cosmological time flow. We are dealing with set of Euclidean  $AdS_4$  with changing scale and coupling along the time flows. The failure of description and the curvature singularity is again due to the weak and strong coupling limit, by which the supergravity description breaks down. But how to understand such a collection of field theories arranged by the cut-off scale and couplings is not clear to us.

Finally, there is the issue of how the holography[26, 27] is realized in the cosmological context[28, 29]. The above cosmology/gauge theory correspondence may serve as a solid framework for the discussion of cosmological holography. In particular the time dependence of the cut-off scale seems important in understanding the number of degrees of the dual gauge theory. Further studies are required in this direction.

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